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On Bounded $m-$Linear Operators in $2-$Normed Spaces

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Abstract

The abstraction of bounded $m-$linear operators in normed spaces is extended in a new direction. Three statements on continuity of the operators are proved to be identical, and the space of operators $(\mathcal{B}(W^m, V), \|.\|)$ is shown to be Banach if $(V, \|.\|)$ is Banach. Moreover, the definition of bounded $m-$linear operators in normed spaces is generalized to $2-$normed spaces and accordingly, we extend related propositions and theorems.

Keywords: Banach space; $2-$normed space; $m-$normed space; $m-$linear operator; continuity.

1 Introduction

Banach [2] first introduced the definition of norm and the concept of normed linear space in 1922. A normed linear space W is a linear space where to each vector $w \in W$, there corresponds a real number $\|w\|$, called norm of w such that;

1. $\|w\| \geq 0$, and $\|w\| = 0$ if and only if $w = 0$.
2. $\|v + w\| \leq \|v\| + \|w\| \; \forall \; v, w \in W$.
3. $\|\alpha w\| = |\alpha|\|w\| \; \forall \; \alpha \in \mathbb{R}$.

The abstraction of linear $2-$normed space is an augmentation of normed linear space. Gähler [4, 5] first brought in the theory of $2-$normed space and $m-$normed space.

W is taken to be a real vector space of dimension greater than 1. A real valued function $\|., .\|$ on $W \times W$, satisfying the conditions below;

1. $\|w, v\| = 0$ if and only if w and v are linearly dependent,
2. $\|w, v\| = \|v, w\|$,
3. $\|\alpha w, v\| = |\alpha|\|w, v\| \;\; \forall \; w, v \in \; W$ and $\alpha \in \mathbb{R}$,
4. $\|w + v, u\| \leq \|w, u\| + \|v, u\| \;\; \forall \; u, v, w \in \; W$,

is called a $2-$norm on W and $(W, \|., .\|)$ is called a linear $2-$normed space. Like norm, $2-$norm is non-negative and $\|v, w + \alpha v\| = \|v, w\|$ for every $v, w \in W$ and $\alpha \in \mathbb{R}$.

$W = \mathbb{R}^2$, together with the Euclidean $2-$norm,

$$\|w_1, w_2\|_E = \text{abs}\left(\begin{vmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{vmatrix}\right),$$

where $w_i = (w_{i1}, w_{i2}) \in \mathbb{R}^2$ for each $i = 1, 2$ is an example of $2-$normed space.

Consider a real vector space W with dim $W \geq m$, m being a positive integer. A real valued function $\|., \dots, .\|$ on W^m satisfying the four conditions below;

1. $\|w_1, \dots, w_m\| = 0$ iff $w_1, \dots, w_m$ are linearly dependent,
2. $\|w_1, \dots, w_m\|$ remains invariant under permutations of $w_1, \dots, w_m$,
3. $\|\alpha w_1, w_2, \dots, w_m\| = |\alpha|\|w_1, \dots, w_m\| \, \forall \, w_1, \dots, w_m \in W$ and $\alpha \in \mathbb{R}$,
4. $\|w_0 + w_1, w_2, \dots, w_m\| \leq \|w_0, \dots, w_m\| + \|w_1, \dots, w_m\|$ for all $w_0, w_1, \dots, w_m \in W$,

is called an $m-$norm on W and the pair $(W, \|., \dots, .\|)$ is called an $m-$normed space.

If W is equipped with an inner product $\langle ., . \rangle$, then,

$$\|w_1, \dots, w_m\|_S := \sqrt{\det[\langle w_i, w_j \rangle]},$$

is a standard $m-$norm on W.

In this work, we redefine the boundedness of $m-$linear operators in a different way and discuss its consequences.

2 Related Works

Gähler [4] first introduced the idea of $2-$norm. The concepts of $2-$metric spaces, linear $2-$normed spaces and their topological properties are discussed in [4, 5]. Diminnie et al. [3] further augmented the work of $2-$normed spaces. Gähler [6, 7] further generalize the concept $2-$metric spaces and $2-$normed spaces to the concept of $m-$metric and $m-$normed spaces. Furthermore, Misiak [12] developed a relationship between $m-$ norm and $m-$inner product. Gunawan and Mashadi [9] showed that every $m-$normed space is an $(m-1)-$normed space.

White Jr. [17] initiated the idea of boundedness in $2-$normed space. Lewandowska [10] introduced the concept of bounded $2-$linear operators on $2-$normed sets. Other than that, Gozali et al. [8] also initiated the idea of bounded $m-$linear functionals in $m-$normed spaces. Abdellaoui and Dahmani [1] studied on convergence in $2-$metric space. Meanwhile, Shaddad et al. [13] studied on semi continuity and fixed points in complete cone metric space. Then, Soenjaya [15] brought in the conception of continuity and boundedness of $m-$linear operators from $m-$normed space to normed space. He introduced a definition of bounded $m-$linear operators as given below.

Let $(W, \|.,\dots,.\|)$ and $(V, \|.\|)$ be respectively an $m-$normed space and a normed space.

Definition 2.1. [15] *An $m-$linear operator $\Lambda : W^m \to V$ is bounded if there exists a $\lambda \in \mathbb{R}$ such that for all $(w_1, w_2, \dots, w_m) \in W^m$,*

$$\|\Lambda(w_1, w_2, \dots, w_m)\| \le \lambda \|w_1, w_2, \dots, w_m\|.$$

Romen and Premjit [11] further augmented the theory of bounded $m-$linear operators. They defined it from a normed space to a normed space.

The definition is

Let $(W, \|.\|)$ and $(V, \|.\|)$ be respectively normed spaces.

Definition 2.2. [11] *An $m-$linear operator $\Lambda : W^m \to V$ is bounded if there exists a $\lambda \in \mathbb{R}$ such that for all $(w_1, w_2, \dots, w_m) \in W^m$,*

$$\|\Lambda(w_1, w_2, \dots, w_m)\| \le \lambda \|w_1\|\|w_2\| \dots \|w_m\|.$$

Further, they introduced a definition on continuity.

Definition 2.3. [11] *An $m-$linear operator $T : W^m \to V$ is continuous at $(w_1, w_2, \dots, w_m) \in W^m$ if for $\epsilon > 0$, there is $\delta > 0$ such that,*

$$\|T(w_1, w_2, \dots, w_m) - T(a_1, \dots, a_m)\| < \epsilon,$$

whenever

$$\|w_1 - a_1\|\|w_2\| \dots \|w_m\| < \delta, \qquad \|a_1\|\|w_2 - a_2\| \dots \|w_m\| < \delta, \quad \dots \quad \text{and}$$
$$\|a_1\|\|a_2\| \dots \|a_{m-1}\|\|w_m - a_m\| < \delta,$$

or

$$\|w_1 - a_1\|\|a_2\| \dots \|a_m\| < \delta, \qquad \|w_1\|\|w_2 - a_2\| \dots \|a_m\| < \delta, \quad \dots \quad \text{and}$$
$$\|w_1\|\|w_2\| \dots \|w_{m-1}\|\|w_m - a_m\| < \delta,$$

where $(a_1, \dots, a_m) \in W^m$.

We may recall the basic definition of $m-$linear operator.

Let V and W be real vector spaces.

Definition 2.4. *An operator $\Lambda : W^m \to V$, that is linear in each variable is called an $m-$linear operator on W.*

Soibam [16] developed recent most concept of $m-$bounded operators that took a significant role in this work. Sharma and Esi [14] did extensive works on convergence in $m-$normed space.

Motivated by the above previous works, we have developed a definition of $m-$ linear operator from a normed space to a normed space. The definition is generalized using the concept of $2-$norm.

3 Method

This study is totally theoretical, and is based on analytical reasoning within the framework of Functional Analysis. The work starts with the introduction of a definition of $m-$linear operators in a different way and consists of theoretical validation of propositions and theorems developed using the newly introduced definition.

We introduce the definition.

Let $(W, \|.\|)$ and $(V, \|.\|)$ be normed spaces.

Definition 3.1. *We say that an $m-$linear operator Λ is bounded if there exists a $\lambda \in \mathbb{R}$ such that for all $(w_1, \ldots, w_m) \in W^m$,*

$$\|\Lambda(w_1, \ldots, w_m)\| \leq \lambda \min_{1\leq i\leq m} \|w_i\|.$$

The present work is totally based on this definition. We developed a necessary and sufficient condition for boundedness of $m-$linear operators from a normed space to a normed space applying the definition of $m-$linearity, Definition 3.1 and triangle inequality. Also, we developed a definition of continuity of $m-$linear operators using the basic concept of continuity of linear operators in Functional Analysis. This is an analogous extension of continuity of $m-$linear operators as given in Definition 2.3. Three equivalent statements on continuity of $m-$linear operators and its boundedness are framed. In order to prove the sufficient condition for the space of $m-$linear operators, we use the concept of inequalities and convergence of sequences in normed spaces. The results obtained are generalized to $2-$normed spaces and we checked its equivalence with the definition of $m-$linear operators in $m-$normed spaces using the concepts of $2-$normed and $m-$normed spaces.

4 Results

From Definition 3.1, we observe that if Λ is bounded and at least one of w_i for $1 \leq i \leq m$ is 0, then $\Lambda(w_1, \ldots, w_m) = 0$.

If Λ is bounded, norm of Λ denoted by $\|\Lambda\|$ is defined as,

$$\|\Lambda\| = \sup_{\|w_i\|\neq 0} \frac{\|\Lambda(w_1,\ldots,w_m)\|}{\min_{1\leq i\leq m}\|w_i\|}. \tag{1}$$

An example of such operator is given below.

Example 4.1. *Let $W = [0,1]$ and $V = \mathbb{R}$, each supplied with the usual norm $\|w\| = |w|$ for all $w \in W, \mathbb{R}$. Define $\Lambda : W^2 \longrightarrow \mathbb{R}$ by $\Lambda(w_1, w_2) = w_1 w_2$. Then, Λ is a bounded $2-$linear operator on W.*

Next, we prove a necessary and sufficient condition of an $m-$linear operator Λ to be a bounded.

Proposition 4.1. *An $m-$linear operator $\Lambda : W^m \to V$ is bounded if and only if, for all $(w_1,\ldots,w_m)$, $(v_1,\ldots,v_m) \in W^m$ and for some constant λ,*

$$\begin{aligned}
&\|\Lambda(w_1,\ldots,w_m) - \Lambda(v_1,\ldots,v_m)\| \\
&\quad\leq \lambda\Bigg(\min_{t_1\in\{w_1-v_1,w_2,\ldots,w_m\}}\|t_1\| + \min_{t_2\in\{v_1,w_2-v_2,\ldots,w_m\}}\|t_2\| \\
&\qquad + \ldots + \min_{t_{m-1}\in\{v_1,v_2,\ldots,w_{m-1}-v_{m-1},w_m\}}\|t_{m-1}\| \\
&\qquad + \min_{t_m\in\{v_1,v_2,\ldots,v_{m-1},w_m-v_m\}}\|t_m\|\Bigg).
\end{aligned}$$

Proof. We suppose that the inequality is true. Setting $(v_1,\ldots,v_m) = (0,\ldots,0)$, we obtain the result.

On the other hand, we suppose that Λ be bounded. Using the definition of $m-$linearity, we obtain

$$\begin{aligned}
&\|\Lambda(w_1,\ldots,w_m) - \Lambda(v_1,\ldots,v_m)\| \\
&\quad= \|\Lambda(w_1-v_1,\ldots,w_m) + \Lambda(v_1,w_2-v_2,\ldots,w_m) + \ldots + \Lambda(v_1,\ldots,v_{m-1},w_m-v_m)\| \\
&\quad\leq \|\Lambda(w_1-v_1,\ldots,w_m)\| + \|\Lambda(v_1,w_2-v_2,\ldots,w_m)\| + \ldots \\
&\qquad + \|\Lambda(v_1,\ldots,w_{m-1}-v_{m-1},w_m)\| + \|\Lambda(v_1,\ldots,v_{n-1},w_m-v_m)\| \\
&\qquad \text{(Using triangle inequality)} \\
&\quad\leq \lambda\Bigg(\min_{t_1\in\{w_1-v_1,w_2,\ldots,w_m\}}\|t_1\| + \min_{t_2\in\{v_1,w_2-v_2,\ldots,w_m\}}\|t_2\| \\
&\qquad + \ldots + \min_{t_{m-1}\in\{v_1,v_2,\ldots,w_{m-1}-v_{m-1},w_m\}}\|t_{m-1}\| + \min_{t_m\in\{v_1,v_2,\ldots,v_{m-1},w_m-v_m\}}\|t_m\|\Bigg).
\end{aligned}$$

□

Definition 4.1. *Let $\Lambda : W^m \to V$ be an $m-$linear operator. We say that Λ is continuous at $(a_1,\ldots,a_m) \in W^m$ if to every $\epsilon > 0$, there is a $\delta > 0$ such that,*

$$\|\Lambda(w_1,\ldots,w_m) - \Lambda(a_1,\ldots,a_m)\| < \epsilon,$$

whenever

$$\begin{aligned}
\min_{t_1\in\{w_1-a_1,w_2,\ldots,w_m\}}\|t_1\| &< \delta, \quad \text{and} \\
\min_{t_2\in\{a_1,w_2-a_2,\ldots,w_m\}}\|t_2\| &< \delta, \quad \text{and} \\
\ldots & \quad \text{and} \\
\min_{t_m\in\{a_1,a_2,\ldots,a_{m-1},w_m-a_m\}}\|t_m\| &< \delta, \quad \text{for all} \quad (w_1,\ldots,w_m) \in W^m.
\end{aligned}$$

Λ is continuous on W^m if it is continuous at each point of W^m.

When $m = 1$, the definition of $m-$linear operator reduces to the usual definition of continuity of a linear operator.

"A linear operator $\Lambda : W \to V$ is continuous at $a \in W$, if to every $\epsilon > 0$, there is a $\delta > 0$ such that $\|\Lambda(w) - \Lambda(a)\| < \epsilon$ for all $w \in W$ satisfying $\|w - a\| < \epsilon$."

Next, we prove that three statements on continuity and boundedness are equivalent. It provides a relationship between the concept of boundedness and continuity.

Theorem 4.1. *For an $m-$linear operator $\Lambda : W^m \to V$, the following statements are equivalent;*

1. *Λ is continuous.*
2. *Λ is continuous at $(0, \ldots, 0) \in W^m$.*
3. *Λ is bounded.*

Proof.

1. We skip the proof for "Statement 1 implies Statement 2".

2. Statement 2 implies Statement 3:
   As Λ be continuous at $(0, \ldots, 0) \in W^m$, we can obtain a $\delta > 0$, such that $\|\Lambda(v_1, \ldots, v_m)\| < 1$ for all $(v_1, \ldots, v_m) \in W^m$, with $\min_{1 \le i \le m} \|v_i\| < \delta$. For $(w_1, \ldots, w_m) \in W^m$, we take

$$v_i = \left(\frac{\delta}{2 \min_{1 \le i \le m} \|w_i\|} \right)^{\frac{1}{m}} w_i.$$

   We obtain

$$\|\Lambda(v_1, \ldots, v_m)\| = \frac{\delta}{2 \min_{1 \le i \le m} \|w_i\|} \|\Lambda(w_1, \ldots, w_m)\|.$$

   It follows that,

$$\|\Lambda(w_1, \ldots, w_m)\| = \frac{2}{\delta} \left(\min_{1 \le i \le m} \|w_i\| \right) \|\Lambda(v_1, \ldots, v_m)\| \le \frac{2}{\delta} \min_{1 \le i \le m} \|w_i\|.$$

   This implies that Λ is bounded, using Definition 3.1.

3. Statement 3 implies Statement 1:
   Since Statement 3 is given, we apply Proposition 4.1. Now, we consider any point $(w_1, \ldots, w_m) \in W^m$. To every $\epsilon > 0$ given, we can obtain a $\delta = \dfrac{\epsilon}{m\|\Lambda\| + 1}$ in such a way that, for every $(v_1, \ldots, v_m) \in W^m$, such that $\min_{t_1 \in \{w_1 - v_1, w_2, \ldots, w_m\}} \|t_1\| < \delta$ and

$\min_{t_2\in\{v_1,w_2-v_2,\dots,w_m\}}\|t_2\|<\delta$ and ... and $\min_{t_m\in\{v_1,v_2,\dots,v_{m-1},w_m-v_m\}}\|t_m\|<\delta$,

$$\begin{aligned}
&\|\Lambda(w_1,\dots,w_m)-\Lambda(v_1,\dots,v_m)\|\\
&\quad\le\|\Lambda\|\Bigg(\min_{t_1\in\{w_1-v_1,w_2,\dots,w_m\}}\|t_1\|+\min_{t_2\in\{v_1,w_2-v_2,\dots,w_m\}}\|t_2\|\\
&\qquad+\dots+\min_{t_{m-1}\in\{v_1,v_2,\dots,w_{m-1}-v_{m-1},w_m\}}\|t_{m-1}\|\\
&\qquad+\min_{t_m\in\{v_1,v_2,\dots,v_{m-1},w_m-v_m\}}\|t_m\|\Bigg)\\
&\quad< m\|\Lambda\|\delta\\
&\quad<\epsilon,\ \text{because}\ \ \delta=\frac{\epsilon}{m\|\Lambda\|+1}.
\end{aligned}$$

Therefore, we come to the point that Λ is continuous, $(w_1,\dots,w_m)\in W^m$ being arbitrary.

□

By $\mathfrak{B}(W^m,V)$, we denote the set of all bounded $m-$linear operators from W^m to V. $\mathfrak{B}(W^m,V)$ is a vector space if the sum $\Lambda_1+\Lambda_2$ of $\Lambda_1,\ \Lambda_2\in\mathfrak{B}(W^m,V)$ is defined by,

$$(\Lambda_1+\Lambda_2)(w_1,\dots,w_m)=\Lambda_1(w_1,\dots,w_m)+\Lambda_2(w_1,\dots,w_m),$$

and the product $\alpha\Lambda$ of $\Lambda\in\mathfrak{B}(W^m,V)$ and a scalar α by,

$$(\alpha\Lambda)(w_1,\dots,w_m)=\alpha\Lambda(w_1,\dots,w_m).$$

Next, we shall attempt to prove that $\mathfrak{B}(W^m,V)$ is a normed space with the norm $\|.\|$ given by (1).

Theorem 4.2. $(\mathfrak{B}(W^m,V),\|.\|)$ *is a normed space where* $\|.\|$ *is defined in* (1).

Proof. We shall prove only that,

$$\|\Lambda\|=\sup_{\|w_i\|\neq0}\frac{\|\Lambda(w_1,\dots,w_m)\|}{\min_{1\le i\le m}\|w_i\|},$$

is a norm.

We take $\|\Lambda\|=0$. Then, $\Lambda(w_1,\dots,w_m)=0$ for $\min_{1\le i\le m}\|w_i\|\neq0$. This is the same as $\Lambda(w_1,\dots,w_m)$ $=0$ for all $(w_1,\dots,w_m)\in W^m$ with each $w_i\neq0$. If Λ is bounded and at least one of w_i for $1\le i\le m$ is 0, $\Lambda(w_1,\dots,w_m)=0$.

So, $\|\Lambda\|=0$ implies and is implied by $\Lambda(w_1,\dots,w_m)=0$ for all $(w_1,\dots,w_m)\in W^m$ i.e., $\|\Lambda\|=0$ implies and is implied by $\Lambda\equiv0$. Other properties are straightforward. It completes the proof. □

Theorem 4.3. $(\mathfrak{B}(W^m,V),\|.\|)$ *is a Banach space if* $(V,\|.\|)$ *is a Banach space.*

Proof. We consider a Cauchy sequence $\langle\Lambda_\lambda\rangle$ in $\mathfrak{B}(W^m,V)$. For a given $\epsilon>0$, we can find $N>0$ such that $\|\Lambda_\lambda-\Lambda_\mu\|<\frac{\epsilon}{2}$ for all $\lambda,\mu>N$. Since $\Lambda_\lambda,\Lambda_\mu\in\mathfrak{B}(W^m,V)$,

$$\|\Lambda_\lambda(w_1,\dots,w_m)-\Lambda_\mu(w_1,\dots,w_m)\|\le\|\Lambda_\lambda-\Lambda_\mu\|\min_{1\le i\le m}\|w_i\|.$$

So, $\|\Lambda_\lambda(w_1,\dots,w_m)-\Lambda_\mu(w_1,\dots,w_m)\| \leq \dfrac{\epsilon}{2}\min_{1\leq i\leq m}\|w_i\|$ for $\lambda,\mu > N$.

Fix $(w_1,\dots,w_m)\in W^m$. Then, the sequence $\langle\Lambda_\lambda(w_1,\dots,w_m)\rangle$ is Cauchy in V. Since $(V,\|.\|)$ is a Banach space, V is complete. So, we can take that $\langle\Lambda_\lambda(w_1,\dots,w_m)\rangle$ is convergent, say to $\Lambda(w_1,\dots,w_m)$. Obviously, Λ is $m-$linear. We shall show that Λ is bounded and Λ_λ is convergent to Λ. For $\lambda > N$,

$$\begin{aligned}\|\Lambda_\lambda(w_1,\dots,w_m)-\Lambda(w_1,\dots,w_m)\| &= \|\Lambda_\lambda(w_1,\dots,w_m)-\lim_{\mu\to\infty}\Lambda_\mu(w_1,\dots,w_m)\|\\ &= \lim_{\mu\to\infty}\|\Lambda_\lambda(w_1,\dots,w_m)-\Lambda_\mu(w_1,\dots,w_m)\|\\ &\leq \frac{\epsilon}{2}\min_{1\leq i\leq m}\|w_i\|.\end{aligned}$$

Hence, $(\Lambda_\lambda-\Lambda)\in\mathfrak{B}(W^m,V)$ for $\lambda > N$. Also, $\Lambda_\lambda\in\mathfrak{B}(W^m,V)$. We can decide

$$\Lambda = \Lambda_\lambda-(\Lambda_\lambda-\Lambda)\in\mathfrak{B}(W^m,V).$$

Taking supremum over all $(w_1,\dots,w_m)\in W^m$ with $\min\limits_{1\leq i\leq m}\|w_i\|\neq 0$, we obtain

$$\|\Lambda_\lambda-\Lambda\| < \epsilon \text{ for } \lambda > N,$$

that is, Λ_λ converges to Λ.

Thus, we can conclude that $(\mathfrak{B}(W^m,V),\|.\|)$ is Banach. $\square$

5 $m-$Linear Operator in a $2-$Normed Space

Let $(W,\|.,.\|)$ be a $2-$normed space and $(V,\|.\|)$ be a normed space.

Definition 5.1. *An $m-$linear operator $\Lambda : W^m \longrightarrow V$ is bounded if there is a $\lambda\in\mathbb{R}$, such that for all $(w_1,\dots,w_m)\in W^m$,*

$$\|\Lambda(w_1,\dots,w_m)\| \leq \lambda \min_{1\leq i,j\leq m, i\neq j}\|w_i,w_j\|.$$

We observe that if Λ is bounded and $w_1,\dots,w_m$ are linearly dependent,

$$\Lambda(w_1,\dots,w_m)=0.$$

If Λ is bounded, we define norm of Λ, denoted by $\|\Lambda\|$ as,

$$\|\Lambda\| = \sup_{\|w_i,w_j\|\neq 0}\frac{\|\Lambda(w_1,\dots,w_m)\|}{\min_{1\leq i,j\leq m}\|w_i,w_j\|}. \tag{2}$$

Note that, whenever we use $\|\Lambda_i,\Lambda_j\|$, it will be understood that $i\neq j$ throughout this section, unless otherwise stated.

Next, we shall attempt to prove a necessary and sufficient condition for Λ to be a bounded $m-$linear operator.

Proposition 5.1. *Let $\Lambda : W^m \to V$ be an $m-$linear operator. Λ is bounded if and only if, for all $(w_1, \dots, w_m), (v_1, \dots, v_m) \in W^m$ and for some constant λ,*

$$\begin{aligned}
&\|\Lambda(w_1, \dots, w_m) - \Lambda(v_1, \dots, v_m)\| \\
&\quad \leq \lambda \Bigg(\min_{t_1,t_2 \in \{w_1 - v_1, w_2, \dots, w_m\}} \|t_1, t_2\| + \min_{t_1,t_2 \in \{v_1, w_2 - v_2, \dots, w_m\}} \|t_1, t_2\| \\
&\qquad + \ldots + \min_{t_1,t_2 \in \{v_1, v_2, \dots, w_{m-1} - v_{m-1}, w_m\}} \|t_1, t_2\| \\
&\qquad + \min_{t_1,t_2 \in \{v_1, v_2, \dots, v_{m-1}, w_m - v_m\}} \|t_1, t_2\| \Bigg).
\end{aligned}$$

Proof. We suppose the inequality is true. For this case, the proof is through by setting

$$(v_1, \dots, v_m) = (0, \dots, 0).$$

For the reverse path, we suppose that Λ is bounded. Applying the definition of $m-$linearity, we obtain

$$\begin{aligned}
&\|\Lambda(w_1, \dots, w_m) - \Lambda(v_1, \dots, v_m)\| \\
&\quad = \|\Lambda(w_1 - v_1, \dots, w_m) + \Lambda(v_1, w_2 - v_2, \dots, w_m) + \ldots + \Lambda(v_1, \dots, v_{m-1}, w_m - v_m)\| \\
&\quad \leq \|\Lambda(w_1 - v_1, \dots, w_m)\| + \|\Lambda(v_1, w_2 - v_2, \dots, w_m)\| + \ldots \\
&\qquad + \|\Lambda(v_1, \dots, w_{m-1} - v_{m-1}, w_m)\| + \|\Lambda(v_1, \dots, v_{m-1}, w_m - v_m)\| \\
&\qquad (\text{ triangle inequality}) \\
&\quad \leq \lambda \Bigg(\min_{t_1,t_2 \in \{w_1 - v_1, w_2, \dots, w_m\}} \|t_1, t_2\| + \min_{t_1,t_2 \in \{v_1, w_2 - v_2, \dots, w_m\}} \|t_1, t_2\| \\
&\qquad + \cdots + \min_{t_1,t_2 \in \{v_1, v_2, \dots, w_{m-1} - v_{m-1}, w_m\}} \|t_1, t_2\| \\
&\qquad + \min_{t_1,t_2 \in \{v_1, v_2, \dots, v_{m-1}, w_m - v_m\}} \|t_1, t_2\| \Bigg).
\end{aligned}$$

□

Definition 5.2. *Let $\Lambda : W^m \to V$ be an $m-$linear operator. Λ is continuous at $(a_1, \dots, a_m) \in W^m$ if, to every $\epsilon > 0$, there is a $\delta > 0$ such that for every $(w_1, \dots, w_m) \in W^m$,*

$$\|\Lambda(w_1, \dots, w_m) - \Lambda(a_1, \dots, a_m)\| < \epsilon,$$

whenever

$$\begin{aligned}
\min_{t_1,t_2 \in \{w_1 - a_1, w_2, \dots, w_m\}} \|t_1, t_2\| &< \delta, && \textit{and} \\
\min_{t_1,t_2 \in \{a_1, w_2 - a_2, \dots, w_m\}} \|t_1, t_2\| &< \delta, && \textit{and} \\
&\ldots && \textit{and} \\
\min_{t_1,t_2 \in \{a_1, a_2, \dots, w_{m-1} - a_{m-1}, w_m\}} \|t_1, t_2\| &< \delta, && \textit{and} \\
\min_{t_1,t_2 \in \{a_1, a_2, \dots, a_{m-1}, w_m - a_m\}} \|t_1, t_2\| &< \delta.
\end{aligned}$$

Theorem 5.1. *Let $\Lambda : W^m \to V$ be an $m-$linear operator;*

1. *Λ is continuous.*
2. *Λ is continuous at $(0, \ldots, 0) \in W^m$.*
3. *Λ is bounded.*

The statements 1, 2 and 3 are equivalent.

Proof.

1. Statement 1 implies Statement 2: This part is skipped.

2. Statement 2 implies Statement 3:
Since Λ is continuous at $(0, \ldots, 0) \in W^m$, we can find a $\delta > 0$ such that $\|\Lambda(v_1, \ldots, v_m)\| < 1$ for all $(v_1, \ldots, v_m) \in W^m$ with $\min_{t_1,t_2 \in \{v_1,\ldots,v_m\}} \|t_1, t_2\| < \delta$. For $(w_1, \ldots, w_m) \in W^m$, we write

$$v_i = \left(\frac{\delta}{2 \min_{t_1,t_2 \in \{w_1,\ldots,w_m\}} \|t_1, t_2\|} \right)^{\frac{1}{m}} w_i.$$

So,

$$\|\Lambda(v_1, \ldots, v_m)\| = \frac{\delta}{2 \min_{t_1,t_2 \in \{w_1,\ldots,w_m\}} \|t_1, t_2\|} \|\Lambda(w_1, \ldots, w_m)\|,$$

and

$$\|\Lambda(w_1, \ldots, w_m)\| = \frac{2}{\delta} \min_{t_1,t_2 \in \{w_1,\ldots,w_m\}} \|t_1, t_2\| \|\Lambda(v_1, \ldots, v_m)\| \leq \frac{2}{\delta} \min_{t_1,t_2 \in \{w_1,\ldots,w_m\}} \|t_1, t_2\|.$$

This implies that Λ is bounded, using Definition 5.1.

3. Statement 3 implies Statement 1:
We are given that Λ is bounded. We shall apply Proposition 5.1. Let $(w_1, \ldots, w_m)$ be any point in W^m. For every $\epsilon > 0$ given, we can find a $\delta = \dfrac{\epsilon}{m\|\Lambda\| + 1}$ such that, for every $(v_1, \ldots, v_m) \in W^m$, with the conditions $\min_{t_1,t_2 \in \{w_1 - v_1, w_2, \ldots, w_m\}} \|t_1, t_2\| < \delta$ and $\min_{t_1,t_2 \in \{v_1, w_2 - v_2, \ldots, w_m\}} \|t_1, t_2\| < \delta$ and $\ldots$ and $\min_{t_1,t_2 \in \{v_1, v_2, \ldots, v_{m-1}, w_m - v_m\}} \|t_1, t_2\| < \delta$,

$$\begin{aligned}
&\|\Lambda(w_1, \ldots, w_m) - \Lambda(v_1, \ldots, v_m)\| \\
&\quad \leq \|\Lambda\| \Bigg(\min_{t_1,t_2 \in \{w_1 - v_1, w_2, \ldots, w_m\}} \|t_1, t_2\| + \min_{t_1,t_2 \in \{v_1, w_2 - v_2, \ldots, w_m\}} \|t_1, t_2\| + \ldots \\
&\qquad + \min_{t_1,t_2 \in \{v_1, v_2, \ldots, w_{m-1} - v_{m-1}, w_m\}} \|t_1, t_2\| + \min_{t_1,t_2 \in \{v_1, v_2, \ldots, v_{m-1}, w_m - v_m\}} \|t_1, t_2\| \Bigg) \\
&\quad < m\|\Lambda\|\delta \\
&\quad < \epsilon, \text{ because } \ \delta = \frac{\epsilon}{m\|\Lambda\| + 1}.
\end{aligned}$$

Therefore, we conclude that Λ is continuous, $(w_1, \ldots, w_m) \in W^m$ being arbitrary.

□

By $\mathfrak{B}(W^m, V)$, we represent the set of all bounded $m-$linear operators from W^m to V.

Theorem 5.2. *$(\mathfrak{B}(W^m, V), \|., .\|)$ is a normed space where $\|., .\|$ is defined in (2).*

Proof. We shall prove only that,

$$\|\Lambda\| = \sup_{\|w_i, w_j\| \neq 0} \frac{\|\Lambda(w_1, \ldots, w_m)\|}{\min_{1 \leq i,j \leq m} \|w_i, w_j\|},$$

is a norm.

If $\|\Lambda\| = 0$, $\Lambda(w_1, \ldots, w_m) = 0$ for $\min_{1 \leq i,j \leq m} \|w_i, w_j\| \neq 0$. This is the same as

$$\Lambda(w_1, \ldots, w_m) = 0,$$

for all $(w_1, \ldots, w_m) \in W^m$ and $w_1, \ldots, w_m$ are linearly independent. If Λ is bounded and $w_1, \ldots, w_m$ are linearly dependent, $\Lambda(w_1, \ldots, w_m) = 0$. Since $\Lambda \in \mathfrak{B}(W^m, V)$, $\|\Lambda\| = 0$ is the same as $\Lambda(w_1, \ldots, w_m) = 0$ for all $(w_1, \ldots, w_m) \in W^m$ i.e., $\|\Lambda\| = 0$ implies and is implied by $\Lambda \equiv 0$.

Proof for other properties are trivial. We can conclude that $(\mathfrak{B}(W^m, V), \|, ., \|)$ is a normed space.

□

Theorem 5.3. *If $(V, \|.\|)$ is a Banach space, $(\mathfrak{B}(W^m, V), \|, ., \|)$ is a Banach space.*

Proof. Consider a Cauchy sequence $\langle \Lambda_\lambda \rangle$ in $\mathfrak{B}(W^m, V)$. To given $\epsilon > 0$, we can find a number $N > 0$ such that $\|\Lambda_\lambda - \Lambda_\mu\| < \dfrac{\epsilon}{2}$ for all $\lambda, \mu > N$. Since $\Lambda_\lambda, \Lambda_\mu \in \mathfrak{B}(W^m, V)$,

$$\|\Lambda_\lambda(w_1, \ldots, w_m) - \Lambda_\mu(w_1, \ldots, w_m)\| \leq \frac{\epsilon}{2} \min_{1 \leq i,j \leq m, i \neq j} \|w_i, w_j\|,$$

for $\lambda, \mu > N$.

We consider a fixed point $(w_1, \ldots, w_m) \in W^m$. The sequence $\langle \Lambda_\lambda(w_1, \ldots, w_m) \rangle$ is Cauchy. Since V is complete, we take that $\langle \Lambda_\lambda(w_1, \ldots, w_m) \rangle$ is convergent, say, to $\Lambda(w_1, \ldots, w_m)$. Clearly, Λ is $m-$linear. Now, we shall attempt to prove that $\Lambda \in \mathfrak{B}(W^m, V)$ and Λ_λ converges to Λ. For $\lambda > N$,

$$\begin{aligned}
\|\Lambda_\lambda(w_1, \ldots, w_m) - \Lambda(w_1, \ldots, w_m)\| &= \|\Lambda_\lambda(w_1, \ldots, w_m) - \lim_{\mu \to \infty} \Lambda_\mu(w_1, \ldots, w_m)\| \\
&= \lim_{\mu \to \infty} \|\Lambda_\lambda(w_1, \ldots, w_m) - \Lambda_\mu(w_1, \ldots, w_m)\| \\
&\leq \frac{\epsilon}{2} \min_{1 \leq i,j \leq m, i \neq j} \|w_i, w_j\|.
\end{aligned}$$

So, $(\Lambda_\lambda - \Lambda) \in \mathfrak{B}(W^m, V)$ for $\lambda > N$. Also, $\Lambda_\lambda \in \mathfrak{B}(W^m, V)$. Now, we can say that
$\Lambda = \Lambda_\lambda - (\Lambda_\lambda - \Lambda) \in \mathfrak{B}(W^m, V)$.

Taking supremum over all $(w_1, \ldots, w_m) \in W^m$ with $\min_{1 \leq i,j \leq m, i \neq j} \|w_i, w_j\| \neq 0$, we obtain

$$\|\Lambda_\lambda - \Lambda\| < \epsilon \text{ for } \lambda > N,$$

implying that Λ_λ converges to Λ.

This completes the proof. □

6 Discussion

When we come from norm to $m-$norm, the boundedness of $m-$linear operators can be defined in various ways. Accordingly, there arise various definitions of continuity of $m-$linear operators.

We have introduced definition of bounded $m-$linear operator $\Lambda : W^m \to V$ in a different pattern, where $(W, \|.\|)$ and $(V, \|.\|)$ are normed spaces i.e. An $m-$linear operator Λ is bounded if there is a constant λ such that for all $(w_1, \ldots, w_m) \in W^m$,

$$\|\Lambda(w_1, \ldots, w_m)\| \leq \lambda \min_{1\leq i\leq m} \|w_i\|.$$

When this definition is generalized to the case of $2-$normed space $(W, \|., .\|)$, it takes the form:

"An $m-$linear operator Λ is bounded if there is a constant λ such that for all $(w_1, \ldots, w_m) \in W^m$,

$$\|\Lambda(w_1, \ldots, w_m)\| \leq \lambda \min_{1\leq i,j\leq m, i\neq j} \|w_i, w_j\|."$$

If it is again generalized to the case of $m-$normed space $(W, \|., \ldots, .\|)$, then the definition is the same as Definition 2.1 as introduced by Soenjaya in [15]. Presently, the research on this area is still ongoing.

Researchers may try new definitions and check whether the definition is equivalent to the existing definitions of $m-$linear operators. The concept of this work may be extended to the concept of fuzzy $m-$linear operators.

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